

Chirality of tensor perturbations for complex values of the Immirzi parameter

Laura Bethke¹ and João Magueijo¹

¹*Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2BZ, United Kingdom*
(Dated: August 4, 2011)

In this paper we generalise previous work on tensor perturbations in a de Sitter background in terms of Ashtekar variables to cover all complex values of the Immirzi parameter γ (previous work was restricted to imaginary γ). Particular attention is paid to the case of real γ . Following the same approach as in the imaginary case, we can obtain physical graviton states by invoking reality and torsion free conditions. The Hamiltonian in terms of graviton states has the same form whether γ has a real part or not; however changes occur for the vacuum energy and fluctuations. Specifically, we observe a γ dependent chiral asymmetry in the vacuum fluctuations only if γ has an imaginary part. Ordering prescriptions also change this asymmetry. We thus present a measurable result for CMB polarization experiments that could shed light on the workings of quantum gravity.

I. INTRODUCTION

Although loop quantum gravity [1–3] is endowed with a rigorous mathematical structure, it is still difficult to obtain GR as a low-energy limit from it and make contact with experiments. However, progress has recently been made on the computation of the graviton propagator [4, 5], and in a previous publication [6] we have identified graviton states within the Hamiltonian framework for a self-dual (or anti-self-dual) connection (for which the Immirzi parameter is $\gamma = \pm i$). The detailed calculation for general imaginary values of γ was provided in [7]. To identify the graviton states that correspond to the dynamical, fluctuating part of space-time we compared our approach to cosmological perturbation theory. After taking several subtleties into account (for more details see [7]) the Ashtekar Hamiltonian indeed reduces on-shell to the standard tensor perturbation Hamiltonian [8]. But novelties come about. We found that only half of the graviton states are physical, retaining only the standard two polarisations for gravitons after reality conditions are imposed. For the physical states we discovered a γ -dependent chirality in the vacuum energy as well as the 2-point function.

In this paper, these results will be generalised to complex γ in a Lorentzian theory. This is a non-trivial algebraic exercise with significant modifications in the results for the intermediate steps, but the final result is remarkably simple. For details on how to derive the second order Hamiltonian for gravitons Reference [7] should be consulted; here we just summarize the framework and highlight the changes that occur for general γ . These are most notably the reality conditions and commutation relations between the canonical variables. It turns out that, in spite of these modifications, the final result is very simple: The vacuum chirality derived in [6, 7] is only present if γ has an imaginary part; for real γ the two graviton polarisations are symmetric.

The plan of the paper is as follows. In Section II we introduce the perturbed metric and connection variables and their classical solution. Section III explains the reality conditions and commutation relations for general γ .

We present a representation of the Hamiltonian in terms of graviton states in section IV. In section V we explain how a complex γ leads to a chirality in the vacuum fluctuations, but only provided that γ has an imaginary part. The special case of real γ will be investigated in section VI. We finish with a concluding section summarising our results.

II. NOTATION AND CLASSICAL SOLUTION

In this Section we lay down the notation, referring the reader to previous publications [6, 7] for details. We consider tensor fluctuations around de Sitter space-time described in the flat slicing, $ds^2 = a^2[-d\eta^2 + (\delta_{ab} + h_{ab})dx^a dx^b]$, where h_{ab} is a symmetric TT tensor, $a = -1/H\eta$, $H^2 = \Lambda/3$ and $\eta < 0$. Using the convention $\Gamma^i = -\frac{1}{2}\epsilon^{ijk}\Gamma^{jk}$ (where Γ^{ab} is the spin connection), the Ashtekar-Immirzi-Barbero connection is given by $A^i = \Gamma^i + \gamma\Gamma^{0i}$, with γ the Immirzi parameter. Making use of the Cartan equations for the zeroth order solution, the canonical variables can be expressed as:

$$A_a^i = \gamma H a \delta_a^i + \frac{a^i}{a} \quad (1)$$

$$E_i^a = a^2 \delta_i^a - a \delta e_i^a, \quad (2)$$

where E_i^a is the densitized inverse triad, canonically conjugate to A_a^i . As in [6, 7] we define δe_a^i via the triad, $e_a^i = a \delta_a^i + \delta e_a^i$; we then raise and lower indices in all tensors with the Kronecker- δ , possibly mixing group and spatial indices. This simplifies the notation and is unambiguous if it's understood that δe is originally the perturbation in the triad. It turns out that δe_{ij} is proportional to the “ v ” variable used by cosmologists [8, 9].

The canonical variables have symplectic structure

$$\{A_a^i(\mathbf{x}), E_j^b(\mathbf{y})\} = \gamma l_P^2 \delta_a^b \delta_j^i \delta(\mathbf{x} - \mathbf{y}) \quad (3)$$

which implies [7]

$$\{a_a^i(\mathbf{x}), \delta e_j^b(\mathbf{y})\} = -\gamma l_P^2 \delta_a^b \delta_j^i \delta(\mathbf{x} - \mathbf{y}) . \quad (4)$$

To make contact with cosmological perturbation theory and standard perturbative quantum field theory we use mode expansions (see [7] for a full explanation):

$$\begin{aligned}\delta e_{ij} &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sum_r \epsilon_{ij}^r(\mathbf{k}) \tilde{e}_{r+}(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\quad + \epsilon_{ij}^{r*}(\mathbf{k}) \tilde{e}_{r-}^\dagger(\mathbf{k}, \eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ a_{ij} &= \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \sum_r \epsilon_{ij}^r(\mathbf{k}) \tilde{a}_{r+}(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \\ &\quad + \epsilon_{ij}^{r*}(\mathbf{k}) \tilde{a}_{r-}^\dagger(\mathbf{k}, \eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\end{aligned}\quad (5)$$

where $\tilde{e}_{rp}(\mathbf{k}, \eta) = e_{rp}(\mathbf{k}) \Psi_e(k, \eta)$ and $\tilde{a}_{rp}(\mathbf{k}, \eta) = a_{rp}(\mathbf{k}) \Psi_a^{rp}(k, \eta)$, and ϵ_{ij}^r are polarization tensors. Amplitudes $\tilde{a}_{rp}(\mathbf{k})$ and $\tilde{e}_{rp}(\mathbf{k})$ have two indices (contrasting with previous literature, e.g. [10, 11]): $r = \pm 1$ for right and left helicities, and p for graviton ($p = 1$) and anti-graviton ($p = -1$) modes. The a_{rp} and e_{rp} can be chosen so as *not* to carry any time-dependence, and for simplicity we will assume that they are equal. After imposing on-shell conditions we'll find that functions $\Psi_a(k, \eta)$ must then carry an r and p dependence.

The classical solution in terms of these variables can be read off from cosmological perturbation theory [7]. Since Ψ_e is proportional to the “ v ” variable used in Cosmology [8, 9], it must satisfy the well-known equation $\Psi_e'' + \left(k^2 - \frac{2}{\eta^2}\right) \Psi_e = 0$ where $'$ denotes derivative with respect to conformal time. This has solution:

$$\Psi_e = \frac{e^{-ik\eta}}{2\sqrt{k}} \left(1 - \frac{i}{k\eta}\right), \quad (6)$$

where the normalization ensures that the amplitudes e_{rp} become annihilation operators upon quantization. The connection can then be inferred from Cartan's torsion-free condition $T^I = de^I + \Gamma_J^I \wedge e^J = 0$. To first order, this is solved by

$$\delta\Gamma_i^0 = \frac{1}{a} \delta e'_{ij} dx^j \quad (7)$$

$$\delta\Gamma_{ki} = -\frac{2}{a} \partial_{[k} \delta e_{i]j} dx^j. \quad (8)$$

These imply $\delta\Gamma^i = \frac{1}{a} \epsilon^{ijk} \partial_j \delta e_{kl} dx^l$, so that

$$a_{ij} = \epsilon_{ikl} \partial_k \delta e_{lj} + \gamma \delta e'_{ij}. \quad (9)$$

Up until this point the calculation is valid for all complex γ . The first novelty in this paper appears upon inserting decomposition (5) into (9), to determine torsion-free conditions in Fourier space. Using relation $\epsilon_{nij} \epsilon_{il}^r k_j = irk \epsilon_{nl}^r$ we obtain:

$$\Psi_a^{r+} = \gamma \Psi_e' + rk \Psi_e \quad (10)$$

$$\Psi_a^{r-} = \gamma^* \Psi_e' + rk \Psi_e, \quad (11)$$

and clearly $\gamma^* = -\gamma$, used in [6], is only valid if γ is imaginary. By writing a generally complex γ as

$$\gamma = \gamma_R + i\gamma_I \quad (12)$$

we find that inside the horizon ($|k\eta| \gg 1$)

$$\Psi_a^{rp} = \Psi_e k (r - i\gamma_R + p\gamma_I), \quad (13)$$

generalizing the expression derived in [7]. We note that the p dependence of these functions only occurs if γ has an imaginary part. For a real γ , Ψ_a is the same for both gravitons and anti-gravitons, as expected (a real connection would be expanded in terms of a single particle \tilde{a}_r , so an index p would be unnecessary; see Section VI for a longer discussion). This is a first hint that the chirality found in [6, 7] is specific to non-real γ .

III. REALITY CONDITIONS AND COMMUTATION RELATIONS

To be able to relate graviton and anti-graviton states (and their respective Hermitian conjugates), we need to impose reality conditions. As in [7], this will be done via the choice of inner product, rather than as operator conditions. Nonetheless it is important to see what these conditions look like in terms of operators (or as classical identities). As the metric is real ($\delta e_{ij} = \delta e_{ij}^\dagger$), we have

$$e_{r+}(\mathbf{k}) = e_{r-}(\mathbf{k}). \quad (14)$$

The definition of the connection implies

$$\Re A^i = \Gamma^i + \gamma_R \Gamma^{0i} \quad (15)$$

$$\Im A^i = \gamma_I \Gamma^{0i}. \quad (16)$$

Compared to the corresponding expressions for imaginary γ (see [7]), we note that the real part of the connection now has a contribution from Γ^{0i} , i.e. the extrinsic curvature. The reality conditions for the connection should embody the non-dynamical torsion-free conditions, i.e. those not involving the extrinsic curvature, which in the Hamiltonian formalism becomes the time derivative of the metric. The full torsion-free conditions representing (9) are now:

$$\begin{aligned}a_{ij} + \bar{a}_{ij} &= 2a (\delta\Gamma_{ij} + \gamma_R \delta\Gamma_{ij}^0) \\ &= 2\epsilon_{ikl} \partial_k \delta e_{lj} + 2\gamma_R \delta e'_{ij}\end{aligned}\quad (17)$$

$$a_{ij} - \bar{a}_{ij} = 2ai\gamma_I \delta\Gamma_{ij}^0 = 2i\gamma_I \delta e'_{ij}, \quad (18)$$

or, in terms of Fourier components:

$$\tilde{a}_{r+}(\mathbf{k}, \eta) + \tilde{a}_{r-}(\mathbf{k}, \eta) = 2rk \tilde{e}_{r+}(\mathbf{k}, \eta) + 2\gamma_R \tilde{e}'_{r+}(\mathbf{k}, \eta) \quad (19)$$

$$\tilde{a}_{r+}(\mathbf{k}, \eta) - \tilde{a}_{r-}(\mathbf{k}, \eta) = 2i\gamma_I \tilde{e}'_{r+}(\mathbf{k}, \eta). \quad (20)$$

Combining (19) and (20) so as to eliminate the time derivative in the metric leads to the condition:

$$i\gamma^* \tilde{a}_{r+}(\mathbf{k}, \eta) - i\gamma \tilde{a}_{r-}(\mathbf{k}, \eta) = 2rk\gamma_I \tilde{e}_{r+}(\mathbf{k}, \eta). \quad (21)$$

Its Hermitian conjugate is:

$$-i\gamma \tilde{a}_{r+}^\dagger(\mathbf{k}, \eta) + i\gamma^* \tilde{a}_{r-}^\dagger(\mathbf{k}, \eta) = 2rk\gamma_I \tilde{e}_{r-}^\dagger(\mathbf{k}, \eta), \quad (22)$$

which also invokes (14). These expressions represent the reality conditions that should be imposed quantum mechanically by the choice of inner product. They are very different from their counterparts for a purely imaginary γ and represent novelty number two in our calculation. For each r and \mathbf{k} there are two independent conditions upon the four operators $a_{rp}(\mathbf{k})$ and $e_{rp}(\mathbf{k})$. In addition to them there is an independent dynamical torsion-free condition. On shell, i.e. using (13) and invoking (14), the connection can be written in terms of the metric according to the weak identity:

$$\begin{aligned}\tilde{a}_{r-}(\mathbf{k}, \eta) &\approx rk\tilde{e}_r + \gamma^* \tilde{e}'_r \rightarrow \tilde{e}_r(r - i\gamma^*)k \\ \tilde{a}_{r+}(\mathbf{k}, \eta) &\approx rk\tilde{e}_r + \gamma \tilde{e}'_r \rightarrow \tilde{e}_r(r - i\gamma)k,\end{aligned}\quad (23)$$

where the latter expression is valid in the limit $k|\eta| \gg 1$. These will be useful in deriving the graviton operators for this theory. They render one of the graviton modes unphysical, fully relating metric and connection.

Before we can set up a quantum theory in terms of graviton operators we need to define the commutation relations in terms of modes. These are obtained, as usual, from the Poisson brackets (3) and (4), leading to:

$$[A_a^i(\mathbf{x}), E_j^b(\mathbf{y})] = i\gamma l_P^2 \delta_a^b \delta_j^i \delta(\mathbf{x} - \mathbf{y}) \quad (24)$$

and

$$[a_a^i(\mathbf{x}), \delta e_j^b(\mathbf{y})] = -i\gamma l_P^2 \delta_a^b \delta_j^i \delta(\mathbf{x} - \mathbf{y}). \quad (25)$$

The commutators for the mode expansions can be derived as in [7] and are:

$$[\tilde{a}_{rp}(\mathbf{k}), \tilde{e}_{sq}^\dagger(\mathbf{k}')] = -i(\gamma_R + p i \gamma_I) \frac{l_P^2}{2} \delta_{rs} \delta_{p\bar{q}} \delta(\mathbf{k} - \mathbf{k}'), \quad (26)$$

where $\bar{q} = -q$. Compared to [7], the factor γp has been replaced by $\gamma_R + p i \gamma_I$. This is algebraic novelty number three, the last one in our calculation. For real γ the p dependence is erased from the commutation relations.

IV. THE HAMILTONIAN

We now have all the ingredients to find a Hamiltonian in terms of graviton creation and annihilation operators (which will be linear combinations of the perturbations in the metric and connection variables). A surprise is in store at this point: in spite of the three novelties in the ingredients, spelled out above, the final result for the graviton operators and Hamiltonian is formally the same.

The gravitational Hamiltonian in terms of Ashtekar variables is given by

$$\begin{aligned}\mathcal{H} = & \frac{1}{2l_P^2} \int d^3x N E_i^a E_j^b \left[\epsilon_{ijk} (F_{ab}^k + H^2 \epsilon_{abc} E_k^c) \right. \\ & \left. - 2(1 + \gamma^2) K_{[a}^i K_{b]}^j \right]\end{aligned}\quad (27)$$

where

$$K_a^i = \frac{A_a^i - \Gamma_a^i(E)}{\gamma} \quad (28)$$

is the extrinsic curvature of the spatial surfaces. The total Hamiltonian includes two further constraints, the Gauss and vector constraint, but they are automatically satisfied by expansions (5) and do not contribute to the order in perturbation theory we will consider [7]. The dynamics of the theory is encoded by the second order Hamiltonian quadratic in first order perturbations. To derive this Hamiltonian, a number of subtleties need to be taken into account which are spelled out in detail in [7]. To write the Hamiltonian as a product of graviton creation and annihilation operators inside the horizon, we need to express the second order Hamiltonian in terms of the mode expansion (5) (see Appendix III of [7]).

We can determine the graviton operators inside the horizon ($|k\eta| \gg 1$) following the same procedure as in [7]. Before reality conditions are imposed there should be unphysical modes that vanish on-shell (and that will turn out to have negative energy and norm). The physical modes should commute with the non-physical modes and reduce, on-shell, to the correct expressions in terms of metric variables. Using these rules, and recalling (23) and (26), we define

$$G_{r\mathcal{P}_+} = \frac{-r}{i\gamma} (\tilde{a}_{r+} - k(r + i\gamma) \tilde{e}_{r+}) \quad (29)$$

$$G_{r\mathcal{P}_-} = \frac{-r}{i\gamma} (\tilde{a}_{r+} - k(r - i\gamma) \tilde{e}_{r+}) \quad (30)$$

$$G_{r\mathcal{P}_+}^\dagger = \frac{r}{i\gamma} (\tilde{a}_{r-}^\dagger - k(r - i\gamma) \tilde{e}_{r-}^\dagger) \quad (31)$$

$$G_{r\mathcal{P}_-}^\dagger = \frac{r}{i\gamma} (\tilde{a}_{r-}^\dagger - k(r + i\gamma) \tilde{e}_{r-}^\dagger). \quad (32)$$

The index $\mathcal{P} = \mathcal{P}_+, \mathcal{P}_-$ denotes physical and unphysical modes, respectively. The normalisation ensures the right behaviour on-shell, i.e. $G_{r\mathcal{P}_-} \approx 0$ and $G_{r\mathcal{P}_+} \approx 2rke_r$. Once the reality conditions (21)–(22)–(14) are enforced one can check that the G^\dagger are indeed the Hermitian conjugate operators of the G . The commutation relations are, as required:

$$[G_{r\mathcal{P}}(\mathbf{k}), G_{s\mathcal{P}}^\dagger(\mathbf{k}')] = \mathcal{P} k l_P^2 \delta_{rs} \delta(\mathbf{k} - \mathbf{k}') \quad (33)$$

$$[G_{r\mathcal{P}_+}(\mathbf{k}), G_{s\mathcal{P}_-}^\dagger(\mathbf{k}')] = 0. \quad (34)$$

These expressions are precisely the same as found in [7] for a purely imaginary γ , in spite of the three algebraic novelties spelled out above. Somehow the modifications conspire to give the same graviton operators and commutators between them. This means that the Hamiltonian in terms of graviton states can be written in the same way as equation (105) of [7]. Just like before an inner product, enforcing the reality conditions, may be found in the representation diagonalizing the G^\dagger operators. The state $\mathcal{P} = \mathcal{P}_+ = 1$ has positive energy and norm, and

$\mathcal{P} = \mathcal{P}_- = -1$ has negative energy and norm. On-shell, the Hamiltonian becomes:

$$\mathcal{H}_{eff}^{ph} \approx \frac{1}{2l_P^2} \int d\mathbf{k} \sum_r [G_r^{ph} G_r^{ph\dagger} (1 + ir\gamma) + G_r^{ph\dagger} G_r^{ph} (1 - ir\gamma)] \quad (35)$$

where $G_r^{ph} = G_{r\mathcal{P}_+}$.

The first term in the Hamiltonian we have just derived (which follows from a EEF ordering) needs to be normal ordered, leading to a chiral (i.e. r -dependent) vacuum energy $V_r \propto 1 + ir\gamma$. The chiral asymmetry is given by

$$\frac{V_R - V_L}{V_R + V_L} = i\gamma. \quad (36)$$

In [7] it was found that for imaginary γ the vacuum energy (VE) is chiral and that for $|\gamma| > 1$ one of the modes has negative VE. This flags a point of interest, since a negative VE is usually associated with fermionic degrees of freedom. We now find that for γ with a real part the VE for each mode is complex. The imaginary part, however, is maximally chiral and so cancels out, when right and left modes are added together. The real part never sees such a cancellation, except in the limit when $|\Im(\gamma)| \rightarrow \infty$, and so the total VE is only zero for the Palatini-Kibble theory.

What is the origin of this result? We already pointed out in [7] that non-perturbatively the Hamiltonian is generally complex, a matter behind many of the novelties we have exposed. On-shell the Hamiltonian is zero and therefore real. The complexity of the Hamiltonian is not to be confused with its Hermiticity after quantization and the inner product should enforce the Hermiticity of the quantum Hamiltonian. Perturbatively, however, the situation is more complicated. As explained in [7], even though the second order Hamiltonian must still be zero on-shell, the portion dependent on first order variables (to be seen as the perturbative Hamiltonian \mathcal{H}^{eff}) evades the Hamiltonian constraint. A number of other novelties of this sort appear when going from the full theory to perturbation theory. It turns out that the classical perturbed Hamiltonian is always real on-shell, even if it's no longer zero. This is still true for a generally complex γ . However, quantum mechanically the perturbative Hamiltonian is only Hermitian, on and off-shell, *if γ is imaginary*. If γ has a real part the normal ordered Hamiltonian is still Hermitian, but the VE is not. This can easily be seen from (35): obviously $G_r^{ph} G_r^{ph\dagger}$ and $G_r^{ph\dagger} G_r^{ph}$ are still Hermitian under the chosen inner product, but their coefficients spoil Hermiticity before, but not after ordering.

What attitude should we take towards this result? One possibility is that there's nothing wrong with it. Obviously the VE couples to the Einstein's equations, but the total is always real. Should we decide, however, that this feature is pathological then there are two possible implications. One is that a purely imaginary γ should be favoured. Another is that a symmetric ordering of the Hamiltonian constraint is to be preferred. For more detail on the different ordering prescriptions see [7]; however it's obvious that EFE or $\frac{1}{2}(EEF + FEE)$ ordering

would satisfy $\mathcal{H} = \mathcal{H}^\dagger$ on and off-shell, before and after ordering. In this case there would be no chirality in the VE; however, as the graviton modes are still the same, the vacuum fluctuations, or the 2-point function, would still exhibit a chiral signature, as investigated in the next section.

V. VACUUM FLUCTUATIONS

As in [7], we now want to compute the 2-point function in terms of connection variables as it determines the vacuum fluctuation power spectrum. This is given by

$$\langle 0 | A_r^\dagger(\mathbf{k}) A_r(\mathbf{k}') | 0 \rangle = P_r(k) \delta(\mathbf{k} - \mathbf{k}'), \quad (37)$$

where $A_r(\mathbf{k})$ represents Fourier space connection variables with handedness r , i.e.

$$A_r(\mathbf{k}) = a_{r+}(\mathbf{k}) e^{-ik \cdot x} + a_{r-}^\dagger(\mathbf{k}) e^{ik \cdot x}. \quad (38)$$

Note that (37) depends on a specific ordering of the 2-point function, and in general we have to consider

$$A^\dagger A \rightarrow \alpha A^\dagger A + \beta A A^\dagger, \quad (39)$$

with $\alpha + \beta = 1$ and $\alpha, \beta > 0$. As (37) is a variance, it must always be real and positive (as opposed to the vacuum energy). Any chiral effects will then leave a measurable imprint on this quantity.

We need to relate the power spectrum to the physical graviton modes found in section IV. This can be done by substituting the on-shell conditions (23) into (29) and (31):

$$a_{r+}^{ph} = \frac{r - i\gamma}{2r} G_{r\mathcal{P}_+} \quad (40)$$

$$a_{r+}^{ph\dagger} = \frac{r + i\gamma^*}{2r} G_{r\mathcal{P}_+}^\dagger \quad (41)$$

$$a_{r-}^{ph} = \frac{r - i\gamma^*}{2r} G_{r\mathcal{P}_+} \quad (42)$$

$$a_{r-}^{ph\dagger} = \frac{r + i\gamma}{2r} G_{r\mathcal{P}_+}^\dagger. \quad (43)$$

Plugging these expressions into (38) we obtain:

$$\begin{aligned} A_r^{ph}(\mathbf{k}) &= \frac{r - i\gamma}{2r} G_{r\mathcal{P}_+}(\mathbf{k}) e^{-ik \cdot x} + \frac{r + i\gamma}{2r} G_{r\mathcal{P}_+}^\dagger(\mathbf{k}) e^{ik \cdot x} \\ A_r^{ph\dagger}(\mathbf{k}) &= \frac{r - i\gamma^*}{2r} G_{r\mathcal{P}_+}(\mathbf{k}) e^{-ik \cdot x} + \frac{r + i\gamma^*}{2r} G_{r\mathcal{P}_+}^\dagger(\mathbf{k}) e^{ik \cdot x} \end{aligned}$$

so that

$$\langle 0 | A_r^{ph\dagger}(\mathbf{k}) A_r^{ph}(\mathbf{k}') | 0 \rangle = P_r(\gamma) \langle 0 | G_{r\mathcal{P}_+}(\mathbf{k}) G_{r\mathcal{P}_+}^\dagger(\mathbf{k}') | 0 \rangle, \quad (44)$$

where

$$P_r(\gamma) = \frac{(r + i\gamma)(r - i\gamma^*)}{4} = \frac{1 - 2\gamma I r + |\gamma|^2}{4}. \quad (45)$$

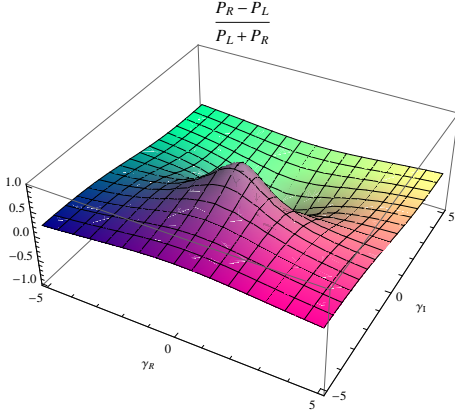


FIG. 1: Power spectrum asymmetry as a function of a generally complex Immirzi parameter γ .

If $\gamma_I r < 0$, $P_r(\gamma)$ is obviously positive. Otherwise,

$$P_r(\gamma) \propto 1 - 2|\gamma_I| + \gamma_I^2 + \gamma_R^2 = (1 - |\gamma_I|)^2 + \gamma_R^2 \quad (46)$$

so this is also positive for any complex γ . Therefore the 2-point function is indeed always real and positive, as required. The chiral asymmetry in the power spectrum can be expressed as

$$\frac{P_R - P_L}{P_R + P_L} = -\frac{2\gamma_I}{1 + |\gamma|^2}, \quad (47)$$

or, for a general ordering,

$$\frac{P_R - P_L}{P_R + P_L} = \frac{2(\beta - \alpha)\gamma_I}{1 + |\gamma|^2}. \quad (48)$$

This implies that for a real γ there is no asymmetry in the vacuum fluctuations for right and left gravitons. The chirality clearly traces to the fact that for an imaginary γ there must exist graviton and anti-graviton modes, i.e. the connection is a complex field. Note however that the presence of a real part of the Immirzi parameter does affect the *absolute* value of the asymmetry due to the factor $|\gamma|$ in the denominator of (47). The power spectrum asymmetry (47) is plotted against a range of values of γ figure (1). It is obviously antisymmetric in γ_I , the minimum and maximum being at $\gamma = \pm i$ respectively which are the values that correspond to a SD/ASD connection. They display the maximum chirality because the Palatini action can naturally be split into a SD and ASD part [3]. The axis $\gamma_I = 0$ corresponds to a real γ and therefore no asymmetry. The chirality also vanishes in the limit $|\gamma| \rightarrow \infty$ which corresponds to the Palatini-Kibble theory.

VI. A PURELY REAL γ

In everything we have derived so far we can take the limit $\Im(\gamma) \rightarrow 0$ and regard the result as the real theory. The question remains as to whether this limit is

the same as a purely real theory, in which all the variables are real from the start. In principle the two might be different, since some aspects of the construction are obviously discontinuous. For example, in a purely real theory expansions (5) have modes a_r and e_r without a p index, so that for a fixed \mathbf{k} and r we start off with two, rather than four modes. It is important to check that this discontinuity does not propagate into our results, leading to expressions different from those taking the limit $\Im(\gamma) \rightarrow 0$ in the complex theory. In this Section we show that this is not the case: at the very least it is possible to set up the real theory so that no discontinuities arise in any of the expressions in this paper, even though there is a jump in the number of independent degrees of freedom. Note that this is far from obvious since the statements $\tilde{e}_{r+} = \tilde{e}_{r-}$ and $\tilde{a}_{r+} = \tilde{a}_{r-}$ are second class constraints in the complex theory, and are not enforced as operator conditions, but as formal conditions on the inner product. The real theory results from imposing them as operator conditions.

Firstly, the commutation relations (26) continuously shrink to

$$[\tilde{a}_r(\mathbf{k}), \tilde{e}_s^\dagger(\mathbf{k}')] = -i\gamma \frac{l_P^2}{2} \delta_{rs} \delta(\mathbf{k} - \mathbf{k}') \quad (49)$$

for a real γ . The reality conditions (21)–(22)–(14) are now trivial (stating $0 = 0$) and do not constrain the theory. However, for graviton operators we can still use definitions (29)–(32) for $G_{r\mathcal{P}}$, simply dropping the p index from their right-hand side, for example:

$$G_{r\mathcal{P}+} = \frac{-r}{i\gamma} (\tilde{a}_r - k(r + i\gamma)\tilde{e}_r). \quad (50)$$

It may appear that we are introducing too many modes. In the complex theory, for a fixed \mathbf{k} and r we start with four modes, $\tilde{a}_{r\mathcal{P}}$ and $\tilde{e}_{r\mathcal{P}}$, from which we build four $G_{r\mathcal{P}}$ and $G_{r\mathcal{P}}^\dagger$. Three reality and torsion-free conditions then reduce them to a single physical operator, as explained after Eqn. (22). For the real theory we only have two modes, a_r and e_r , from which we build four $G_{r\mathcal{P}}$ and $G_{r\mathcal{P}}^\dagger$ without having any reality conditions. However, upon closer inspection we see that for fixed \mathbf{k} and r there are only two independent modes among the $G_{r\mathcal{P}}$ and $G_{r\mathcal{P}}^\dagger$: In the complex theory we needed the reality conditions to ensure that the $G_{r\mathcal{P}}^\dagger$ were in fact the Hermitian conjugates of the $G_{r\mathcal{P}}$. If we drop the index p from their expressions, as in (50), then this fact follows trivially from their definitions and the linearity of the \dagger operation. Hence by defining gravitons operators in the real theory we do preserve the number of independent degrees of freedom.

The issue persists on how to eliminate the non-physical mode. This is done by imposing the torsion-free condition, relating the a_r to the e_r , which amounts to disqualifying the $G_{r\mathcal{P}-}$ mode. A *possible* implementation, even in the real theory, is to do this via the inner product. As in [7], we work in a holomorphic representation which

diagonalizes $G_{r\mathcal{P}}^\dagger$, i.e.: $G_{r\mathcal{P}}^\dagger \Phi(z) = z_{r\mathcal{P}} \Phi(z)$. Then (33) implies:

$$G_{r\mathcal{P}} \Phi = \mathcal{P} k l_P^2 \frac{\partial \Phi}{\partial z_{r\mathcal{P}}} . \quad (51)$$

Following the same procedure as in [7] we find

$$\langle \Phi_1 | \Phi_2 \rangle = \int dz d\bar{z} e^{\mu(z, \bar{z})} \bar{\Phi}_1(\bar{z}) \Phi_2(z) \quad (52)$$

with:

$$\mu(z, \bar{z}) = \int d\mathbf{k} \sum_{r\mathcal{P}} \frac{-\mathcal{P}}{k l_P^2} z_{r\mathcal{P}}(\mathbf{k}) \bar{z}_{r\mathcal{P}}(\mathbf{k}) , \quad (53)$$

rendering the states built from operators with $\mathcal{P} = \mathcal{P}_- = -1$ non-normalizable. As long as this procedure is adopted for the real theory the expressions found in this paper are continuous, and the limit $\Im(\gamma) \rightarrow 0$ does indeed represent the real theory.

VII. CONCLUSION

In this paper we have generalized the results of [7] to cover all values of the Immirzi parameter. Our analysis

shows that an imaginary part of γ is needed to produce a chiral effect in the vacuum fluctuations, whereas a purely real γ would give the same physical Hamiltonian for right- and left-handed gravitons. The greatest asymmetry occurs for the values $\gamma = \pm i$, corresponding to a SD/ASD connection and the subject of [6]. Here, as in previous work, the chirality also depends on the ordering used for the 2-point function. Although this implies that an observation of this asymmetry cannot be traced back to one single cause, it is still a striking prediction of quantum gravity in the Ashtekar formalism.

It was shown in [12] that even a small chiral effect in the gravitational wave background would greatly simplify its detection, making us hopeful that a test of our prediction could even be achieved by PLANCK. Note that other mechanisms exist that produce a similar chiral effect [13–15], but the one pointed out here is by far the simplest. It would be interesting to make contact with the work of [4], where a chiral contribution was found for the graviton propagator. However, in this publication a Euclidean signature and a real γ were used, basically the opposite of our set-up, making the link between the predictions unclear.

Acknowledgements We thank Dionigi Benincasa, Gianluca Calcagni and Chris Isham for help regarding this project.

-
- [1] R. Gambini and J. Pullin, *Loops, Knots, Gauge theories and Quantum gravity*, CUP, Cambridge 1996.
 - [2] C. Rovelli, *Quantum Gravity*, CUP, Cambridge, 2004.
 - [3] T. Thiemann, *Modern Canonical Quantum General Relativity*, CUP, Cambridge, 2007.
 - [4] C. Rovelli, *Phys. Rev. Lett.* 97, 151301, 2006; E. Bianchi et al, *Class. Quant. Grav.* 23, 6989, 2006; E. Bianchi, E. Magliaro, C. Perini, *Nuc. Phys. B* 822, 245, 2009.
 - [5] C. Rovelli, arXiv:1004.1780 and 1010.1939.
 - [6] J. Magueijo and D. Benincasa, *Phys. Rev. Lett.* 106: 121302, 2011.
 - [7] L. Bethke and J. Magueijo, arXiv:1104.1800.
 - [8] V. Mukhanov, “Physical Foundations of cosmology”, CUP, Cambridge 2005.
 - [9] A. Liddle and D. Lyth, “Cosmological Inflation and Large-scale Structure”, CUP, Cambridge 2000.
 - [10] A. Ashtekar, C. Rovelli and L. Smolin, *Phys. Rev. D* 44, 1740, 1991.
 - [11] L. Freidel and L. Smolin, *Class. Quant. Grav.* 21: 3831-3844, 2004.
 - [12] C. Contaldi, J. Magueijo and L. Smolin, *Phys. Rev. Lett.* 101: 141101, 2008.
 - [13] S. Alexander, arXiv:0706.4481.
 - [14] S. Alexander and G. Calcagni, *Found. Phys.* 38, 1148-1184, 2008; *Physics Letters B* 672 (2009) 386.
 - [15] S. Mercuri, arXiv:1007.3732.